§ Lecture 18

13-11-18

We consider: $\vec{y}'(t) = P(t)\vec{y}(t) + \vec{r}(t)$

fact: Assume $\Phi(t)$ fundamental matrix for $\vec{y}'(t) = P(t)\vec{y}(t)$

then general solution: $\vec{y}(t) = \Phi(t)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \vec{Y}(t)$

particular solution

Moral: With $\Phi(t)$, we need to find a paticular sol. $\vec{Y}(t)$.

For equation with constant coefficient:

$$\vec{y}'(t) = A\vec{y}(t) + \vec{r}(t). \quad ---- (**)$$

We make the transformation $A = QJQ^{-1}$

Jordan form.

$$\underbrace{(Q^{-1}\vec{y})'(t)}_{\vec{z}'(t)} = J\underbrace{(Q^{-1}\vec{y}(t))}_{\vec{z}(t)} + \underbrace{Q^{-1}\vec{r}(t)}_{\vec{g}(t)}$$

$$\Rightarrow \quad \vec{z}'(t) = J\vec{z}(t) + \vec{g}(t).$$

$$\because \quad (e^{-Jt}\vec{z}(t))' = -e^{-Jt}J\vec{z}(t) + e^{-Jt}\vec{z}'(t)$$

$$(e^{-Jt}\vec{z}(t))' = e^{-Jt}\vec{g}(t)$$

solution to homogeneous eqn.

$$\rightsquigarrow \quad \vec{z}(t) = e^{Jt}\left(\int e^{-Jt}\vec{g}(t) + \vec{c}\right)$$

$$\boxed{\text{Caution: } e^{tA}\cdot e^{tB} \neq e^{t(A+B)} \text{ unless } AB=BA}$$

Caution: If we have $A(t) \in M_{n\times n}(\mathbb{R})$.

- $\frac{d}{dt}(e^{A(t)}) \neq e^{A(t)} \cdot A(t)$

$\because \frac{d}{dt}(A(t))^k = \frac{dA}{dt} A^{n-1} + A \frac{dA}{dt} A^{n-2} + \dots + A^{n-1}\frac{dA}{dt}$

as $A\frac{dA}{dt} \neq \frac{dA}{dt} A$ in general.

- In the case $A(t) = t \cdot A$, then we have $\frac{dA(t)}{dt} = A$ which allows us to do $\frac{d}{dt}(e^{tA}) = Ae^{tA} = e^{tA}A$.

Eg.

$$\vec{y}'(t) = \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}_{=A} \vec{y}(t) + \underbrace{\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}}_{=g(t)}$$

Step 1: $P_A(x) = \det(A - xI) = \det\begin{pmatrix} -2-x & 1 \\ 1 & -2-x \end{pmatrix}$

$= x^2 + 4x + 3$

$\Rightarrow$ Two distinct eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$.

Step 2: **Find Q:** solve for eigenvectors

$$A+I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \rightsquigarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A+3I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- $$Q^{-1}AQ = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} = J$$

Let $\vec{z}(t) = Q^{-1}\vec{y}(t)$,

$$\vec{g}(t) = Q^{-1}\vec{r}(t) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2e^{-t}+3t \\ 2e^{-t}-3t \end{pmatrix}.$$

- $$e^{-Jt}\vec{g}(t) = \frac{1}{2}\begin{pmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 2e^{-t}+3t \\ 2e^{-t}-3t \end{pmatrix} = \begin{pmatrix} 1+\frac{3}{2}te^{t} \\ e^{2t}-\frac{3}{2}te^{3t} \end{pmatrix}$$

Now we solve:

$$\vec{z}_p(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \cdot \left[\int \begin{pmatrix} 1+\frac{3}{2}te^{t} \\ e^{2t}-\frac{3}{2}te^{3t} \end{pmatrix} dt + \cancel{\vec{c}} \right]$$

0: homogeneous solutions.

$$= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \cdot \left[\begin{pmatrix} t \\ 0 \end{pmatrix} + \frac{1}{2}e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{3}{2}(te^{t}-e^{t})\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\left(te^{3t}-\frac{1}{3}e^{3t}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cancel{\vec{c}} \right]$$

0.

$$= te^{-t}\begin{pmatrix}1\\0\end{pmatrix} + e^{-t}\begin{pmatrix}0\\ \frac{1}{2}\end{pmatrix} + t\begin{pmatrix}\frac{3}{2}\\ -\frac{1}{2}\end{pmatrix} + \begin{pmatrix}-\frac{3}{2}\\ \frac{1}{6}\end{pmatrix}$$

Finally: A particular solution $\vec{Y}(t)$

$$= \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\left(z_p(t)\right)$$

$$= \begin{pmatrix}1\\1\end{pmatrix} te^{-t} + \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}e^{-t} + \begin{pmatrix}1\\2\end{pmatrix}t + \begin{pmatrix}-\frac{4}{3}\\ -\frac{5}{3}\end{pmatrix}$$

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§ Variation of parameter:

- Suppose we have a general 1^{st} order linear system:

$$\vec{y}\,'(t) = P(t)\vec{y}(t) + \vec{r}(t).$$

Given $\Phi(t)$ for $\Phi'(t) = P(t)\Phi(t)$ fundamental matrix

Q: How to find a particular solution?

Idea: $\frac{d}{dt}\left(\Phi^{-1}(t)\,\vec{y}(t)\right) = \frac{d(\Phi^{-1})}{dt}\vec{y} + \Phi^{-1}\frac{d\vec{y}}{dt}$

$$\boxed{\frac{d(\Phi^{-1})}{dt} = -\Phi^{-1}\frac{d\Phi}{dt}\Phi^{-1}}$$

$$= -\Phi^{-1}\frac{d\Phi}{dt}\Phi^{-1}\vec{y} + \Phi^{-1}\frac{d\vec{y}}{dt}$$

$$= -\Phi^{-1}P(t)\Phi\Phi^{-1}\vec{y} + \Phi^{-1}\frac{d\vec{y}}{dt}$$

$$= \Phi^{-1}\left(\frac{d\vec{y}}{dt} - P(t)\vec{y}\right)$$

$\Rightarrow$ If we let $\vec{z}(t) = \Phi(t)^{-1}\vec{y}(t)$

Then we get: $\vec{z}'(t) = \Phi^{-1}(t)\vec{r}(t)$

$$\rightsquigarrow \quad \vec{z}(t) = \left[\int \Phi^{-1}(t)\,\vec{r}(t)\,dt + \vec{c}\,\right]$$

$$\Rightarrow \boxed{\vec{y}(t) = \Phi(t)\left[\int \Phi^{-1}(t)\,\vec{r}(t)\,dt + \vec{c}\,\right]}$$

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§ Method of undetermined coefficient:

For inhomogenous equation with constant coefficient:

$$\vec{y}'(t) = A\vec{y}(t) + \vec{r}(t) \quad \cdots\cdots \quad (*)$$

$$\vec{r}(t) = \begin{pmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{pmatrix}$$

Assume: $\vec{r}(t) = \vec{P}_k(t) = \vec{v}_0 + \vec{v}_1 t + \cdots + \vec{v}_k t^k$

or $e^{\alpha t} P_k(t)$ or $e^{\alpha t}\cos\mu t\, P_k(t)$ or $e^{\alpha t}\sin\mu t\, P_k(t)$ or

a summation of terms of these form

Rk: We can Assume: $\vec{r}(t) = \sum \vec{g}_j(t)$

superposition principle $\rightarrow$ we can solve for $\vec{Y}_j$ s.t.

$$\vec{Y}_j'(t) = A\,\vec{Y}_j(t) + \vec{g}_j(t)$$

and take $\vec{Y} = \sum_j Y_j$ which is a particular sol

$$\vec{Y}'(t) = A\vec{Y}(t) + \vec{r}(t)$$

Case 1: $\vec{F}(t) = P_k(t)$

Guess: $\vec{Y}(t) = Q_\ell(t) = \vec{u}_0 + \cdots + \vec{u}_\ell t^\ell$.

$$\vec{Y}'(t) = \vec{u}_1 + 2\vec{u}_2 t + \cdots + j\vec{u}_j t^{j-1} + \cdots + \ell \vec{u}_\ell t^{\ell-1}$$

Claim: if s = alg. mult. of 0 in $P_A(x)$

(if 0 is NOT an eigenvalue $\Rightarrow s = 0$).

then $\vec{Y}(t)$ can be solved with $\ell = k + s$

Plug in: $\vec{Y}'(t) = A\vec{Y}(t) + P_k(t)$

$$(\vec{u}_1 - A\vec{u}_0) + (2\vec{u}_2 - A\vec{u}_1)t + \cdots + ((j+1)\vec{u}_{j+1} - A\vec{u}_j)t^j + \cdots + A\vec{u}_\ell t^\ell$$

$$= \vec{v}_0 + \vec{v}_1 t + \cdots + \vec{v}_k t^k$$

Equation to solve:

$A = QJQ^{-1}$, $\vec{z}_i = Q^{-1}\vec{u}_i$, $\vec{w}_i = Q^{-1}\vec{v}_i$

$$\left.\begin{aligned} A\vec{u}_\ell &= 0 \\ \ell\vec{u}_\ell - A\vec{u}_{\ell-1} &= 0 \\ &\vdots \\ (k+1)\vec{u}_{k+1} - A\vec{u}_k &= \vec{v}_k \\ &\vdots \\ \vec{u}_1 - A\vec{u}_0 &= \vec{v}_0 \end{aligned}\right\} \qquad \begin{aligned} J\vec{z}_\ell &= 0 \\ \ell\vec{z}_\ell - J\vec{z}_{\ell-1} &= 0 \\ &\vdots \\ (k+1)\vec{z}_{k+1} - J\vec{z}_k &= \vec{w}_k \\ &\vdots \\ \vec{z}_1 - J\vec{z}_0 &= \vec{w}_0 \end{aligned}$$

Eg 1: Say alg. mult. of 0 for $A = 0$

i.e. J is invertible $\Longrightarrow$ the equation can be solved for $k = l$.

Eg 2: Let $J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ a Jordan block.

$$\mathrm{Im}(J) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \mathrm{Ker}(J) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\mathrm{Im}(J^2) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}. \quad \text{and } J^3 = 0.$$

Idea:

- $\vec{z}_1 - \cancelto{0}{J\vec{z}_0} = \vec{w}_0$

  $\uparrow$ choose $\vec{z}_1 = \vec{w}_0$

- $2\vec{z}_2 - J\vec{z}_1 = \vec{w}_1 \rightsquigarrow$ let $\vec{z}_2 = \frac{1}{2}(J\vec{z}_1 + \vec{w}_1)$

- we have solved $\vec{z}_{k+1}$, then we want

  $(k+2)\vec{z}_{k+2} - J\vec{z}_{k+1} = 0. \Longrightarrow \vec{z}_{k+2} \in \mathrm{Im}(J)$

- similarly, $\vec{z}_{k+3} \in \mathrm{Im}(J^2). \Rightarrow \boxed{J\vec{z}_{k+3} = 0}$

  i.e. it can be solved for $l = 3$.

Case 2: $\vec{r}(t) = e^{\alpha t} P_k(t)$, then we let $Y(t) = e^{\alpha t} Q_l(t)$.

$$\left(\frac{d}{dt} e^{\alpha t}\right)(Q_l(t)) = e^{\alpha t} A Q_l(t) + e^{\alpha t} P_k(t).$$

$$\rightsquigarrow \quad \left(\frac{d}{dt} + \alpha I\right)(Q_l(t)) = A\, Q_l(t) + P_k(t).$$

$$\Longleftrightarrow \quad \frac{d}{dt}(Q_l(t)) = (A - \alpha I) Q_l(t) + P_k(t).$$